

# VANISHING HOMOLOGY

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**ABSTRACT.** In this paper we introduce a new homology theory devoted to the study of families such as semi-algebraic or subanalytic families and in general to any family definable in an o-minimal structure (such as Denjoy-Carleman definable or  $\ln - \exp$  definable sets). The idea is to study the cycles which are vanishing when we approach a special fiber. This also enables us to derive local metric invariants for germs of definable sets. We prove that the homology groups are finitely generated.

## 0. INTRODUCTION

The description of the topology of a set nearby a singularity is a primary focus of attention of algebraic geometers. We can regard a semi-algebraic singular subset of  $\mathbb{R}^n$  as a metric subspace. Then the behavior of the metric structure of a collapsing family reflects implicit information on the geometry of the singularity of the underlying set which is much more accurate than the one provided by the study of the topology.

In [V1], the author proved a bi-Lipschitz version of Hardt's theorem [H]. This theorem pointed out that semi-algebraic bi-Lipschitz equivalence is a good notion of equisingularity to classify semi-algebraic subsets from the metric point of view. For this purpose, it is also very helpful to find invariants such as homological invariants.

In this paper we introduce a homology theory for families of subsets which provides information about the behavior of the metric structure of the fibers when we approach a given fiber. This enables us to construct local metric invariants for singularities. We prove that these homology groups are finitely generated when the family is definable in an o-minimal structure. This allows, for instance, to define an Euler characteristic which is a metric invariant for germs of algebraic or analytic sets.

In [GM], M. Goresky and R. MacPherson introduced intersection homology and showed that their theory satisfies Poincaré duality for pseudo-manifolds which cover a quite large class of singular sets and turned out to be of great interest. They also managed to compute the intersection homology groups from a triangulation which yields that they are finitely generated. In [BB1] L. Birbrair and J.-P. Brasselet define their admissible chains to construct the metric homology groups. Both theories select some chains by putting conditions on the support of the chains. Our approach is similar in the sense that our homology groups will depend on a *velocity* which estimates the rate of vanishing of the support of the chains.

Our method relies on the result of [V1], where the author showed existence of a triangulation enclosing the metric type of a definable singular set. To compute the vanishing homology groups we will not use the triangulation constructed in [V1] but Proposition

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3.2.6 of the latter paper (which was actually the main step of the construction). It makes it possible for the results proved below to go over non necessarily polynomially bounded o-minimal structures. It seems that the method of the present paper could be generalized to prove that the metric homology groups introduced in [BB1] are finitely generated as well.

It is well known that, given a definable family, we may always study the evolution of the fibers by studying what is called by algebraic geometers “the generic fiber” (see example 1.3.2 for a precise definition).

Therefore if we carry out a homology theory for definable subsets in an o-minimal structure expanding a given arbitrary real closed field, we will have a homology theory for families. This is the point of view of the present paper. Hence, even for families of subsets of  $\mathbb{R}^n$ , the case of an arbitrary real closed field will be required. Our approach will be patterned on the one of the classical homology groups as much as possible. Some statements (Theorem 3.2.2) are close to those given by Goresky and MacPherson for intersection homology but of course the techniques are radically different since the setting is not the same.

The admissible chains depend on a velocity which is a convex subgroup  $v$  of our real closed field  $R$ . For instance, if  $R$  is the field of real algebraic Puiseux series endowed with the order making the indeterminate  $T$  smaller than any positive real number,  $v$  may be the subgroup

$$(0.1) \quad \{x : \exists N \in \mathbb{N}, |x| \leq NT^2\}.$$

The *v-admissible chains* are the chains having a “v-thin” support. Roughly speaking, if  $v$  is as above, *v-thin* subsets of  $R^n$  are the generic fibers of families of sets whose fibers collapse onto a lower dimensional subset with at least the velocity  $Nt^2$  (if  $t$  is the parameter of the family,  $N \in \mathbb{N}$ ). For instance, let us consider the cycle given by Birbrair and Goldshtein’s example. Namely, the subset of  $X \subset R^4$  defined by:

$$(0.2) \quad \begin{aligned} x_1^2 + x_2^2 &= T^{2p}, \\ x_3^2 + x_4^2 &= T^{2q}. \end{aligned}$$

This set is the generic fiber of a family of tori, such that the support of the generators of  $H_1(X)$  collapse onto a point at rate  $T^p$  and  $T^q$  respectively. Therefore, if for instance  $p = 0$  and  $q = 2$  then the 0-fiber is a circle and this family of torus is *v-thin* (with  $v$  like in (0.1)).

Taking all the *v-admissible* chains of a definable set  $X$ , we get a chain complex which immediately gives rise to the *v-vanishing homology groups*  $H_j^v(X)$ . We will show that these groups are finitely generated (Corollary 3.2.3).

If  $X$  is the set defined by (0.2) with  $v$  like in (0.1), the *v-vanishing* homology groups depend on  $p$  and  $q$ . For instance, we will prove (see Example 5.2.2) that if  $p = 0$  and  $q = 2$ :

$$H_1^v(X) = \mathbb{Q}$$

(if  $\mathbb{Q}$  is our coefficient group), and  $H_2^v(X) = \mathbb{Q}$ .

We may summarize it by saying that we get all the  $T^2$ -thin cycles of  $X$ . The group  $H_j^v(X)$  is not always a subgroup of  $H_j(X)$ . In general we may also have cycles that do not appear in the classical homology groups, i. e. which are in the kernel of the natural map

$H_j^v(X) \rightarrow H_j(X)$ . The following picture illustrates an example for which such a situation occurs:

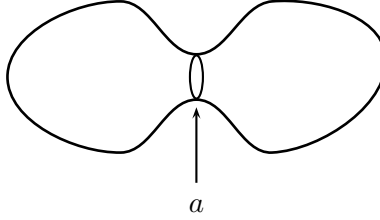


FIGURE 1.

The cycle  $a$  is collapsing onto a point faster than the set itself is collapsing. We see that we have an admissible one dimensional chain  $a$  which bounds a two dimensional chain which may fail to be admissible (depending on the velocity  $v$ ). Therefore  $H_1^v(X) \neq 0$  (while  $H_1(X) = 0$ ).

This homology theory is not a homotopy invariant. It is preserved by Lipschitz homotopies but these are very hard to construct. For instance, given a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  it is well known that there exists a topological deformation retract of  $f^{-1}(0; \varepsilon)$  onto  $f^{-1}(0)$ . It is easy to see that it is *not* possible to find such a retract which would be Lipschitz if  $f(x; y) = y^2 - x^3$ . The method used in this paper provides homotopies that are not Lipschitz but which preserve admissible chains. It seems that one could define various homology theories for which this method could be adapted. The theory developed below seemed to the author the simplest one and the most natural to start.

We compute the vanishing homology groups in terms of some basic sets obtained by constructing some nice cells decompositions (Theorem 3.2.2). For this we construct a homotopy which carries a given singular chain to a chain of these basic sets (Proposition 3.2.1). The homotopy has to preserve thin subsets. We are not able to construct such a homotopy for any admissible chain. Chains for which we can construct such a homotopy are called strongly admissible and are chains for which the distances in the support are known in a very explicit way. Therefore, the first step is to show that any class in  $H_j^v(X)$  has a strongly admissible representant (Lemma 3.1.3). This is achieved by constructing some rectilinearizations of  $v$ -thin sets (Proposition 2.2.4). These are maps which transform our set into a union of hyperplanes crossing normally while controlling the distances in the transformation.

A non trivial convex subgroup  $v$  may be regarded as an interval in  $R$  which has no endpoint. This fact will somewhat complicate our task. To overcome this difficulty, we introduce an extra point  $u$  “at the end of  $v$ ” which will fill the gap. This point living in an extension  $k_v$  of  $R$ , we will carry out most of the constructions rather in  $k_v$  than in  $R$ . The precise definition of  $k_v$  and the basic related notions are provided in the first section below. An advantage of using model theory is that we are able to carry out the theory for all the possible velocities (see example 1.1.2) in the same time.

We may use these homology groups to derive invariants for semi-algebraic singularities. Given a germ  $A$  of semi-algebraic subset of  $\mathbb{R}^n$  at the origin, the link of  $A$  is the subset

$$L_r := A \cap B(0; r)$$

for  $r$  small enough. It is known that the homology of the latter set is a topological invariant of  $A$ . The cycles of  $L_r$  are collapsing to a single point with a certain “rate”. This rate is related to the metric type of the singularity.

It is proved in [V2] that the metric type of the generic fiber of the family  $L_r$ , namely  $L_{0+}$ , is a metric invariant of  $A$ . Therefore the vanishing homology groups  $H_j^v(L_{0+})$  are semi-algebraic bi-Lipschitz invariants of  $A$  (see section 4.).

**Content of the paper.** In section 1, we provide all the basic definitions about the vanishing homology. We prove in the next section some cell decomposition theorems and rectilinearization theorems necessary to compute the vanishing homology groups. In section 3, we compute the  $v$ -vanishing homology groups in terms of this cell decomposition. The main result is Theorem 3.2.2 which yields that the homology groups are finitely generated. In section 4 we give an application: we find local metric invariants for singularities. The last section computes the vanishing homology groups on some examples.

The reader is referred to [C] or [vD] for basic facts about o-minimal structures.

**Notations and conventions.** Throughout this paper we work with a fixed o-minimal structure expanding a real closed field  $R$ . Let  $\mathcal{L}_R$  be the first order language of ordered fields together with an  $n$ -ary function symbol for each function of the structure. The word definable means  $\mathcal{L}_R$ -definable. The language  $\mathcal{L}_R(u)$  is the language  $\mathcal{L}_R$  extended by an extra symbol  $u$ .

The letter  $G$  will stand for an abelian group (our coefficient group). *Singular simplices* will be definable continuous maps  $c : T_j \rightarrow X$ ,  $T_j$  being the  $j$ -simplex spanned by  $0, e_1, \dots, e_j$  where  $e_1, \dots, e_j$  is the canonical basis of  $R^j$ . Sometimes, we will work in an extension  $k_v$  of  $R$  and simplices will actually be maps  $c : T_j(k_v) \rightarrow k_v^n$  where  $T_j(k_v)$  is the extension of  $T_j$  to  $k_v$ . Given a definable set  $X \subset R^n$  we denote by  $C(X)$  the chain complex of definable chains with coefficients in a given group  $G$ . We will write  $|c|$  for the support of a chain  $c$ .

By **Lipschitz function** we will mean a function  $f$  satisfying

$$|f(x) - f(x')| \leq N|x - x'|$$

for some integer  $N$ . It is important to notice that we require the constant to be an integer for  $R$  is not assumed to be archimedean. A map  $h : A \rightarrow R^n$  is Lipschitz if all its components are, and a homeomorphism  $h$  is bi-Lipschitz if  $h$  and  $h^{-1}$  are Lipschitz.

We denote by  $\pi_n : R^n \rightarrow R^{n-1}$  the canonical projection and by  $cl(X)$  the closure of a definable set  $X$ .

## 1. DEFINITION OF THE VANISHING HOMOLOGY.

**1.1. The velocity  $v$ .** We shall use some very basic facts of model theory. We refer the reader to [M] for basic definitions.

The vanishing homology depends on a **velocity**  $v$  which estimates the rate of vanishing of the cycles. This is a convex subgroup  $v$  of  $(R; +)$  (convex in the sense that it is a convex subset of  $R$ ).

We then define a 1-type by saying that a sentence  $\psi(u) \in \mathcal{L}_R(u)$  is in this type iff the set

$$\{x \in R : \psi(x)\}$$

contains an interval  $[a; b]$  with  $a \in v$  and  $b \notin v$ . This type is complete due to the o-minimality of the theory.

We will denote by  $k_v$  an  $\mathcal{L}_R$ -elementary extension of  $R$  realizing this type.

Roughly speaking we can say that the velocity is characterized by a cut in  $R$ , at which the gap is “bigger” than the distance to the origin. This is to ensure that the sum of two admissible chains will be admissible (see section 1.3).

**Notations.** Throughout this paper, a velocity  $v$  is fixed and  $u$  is the point realizing the corresponding type in  $k_v$ .

We define a convex subgroup  $w$  of  $(k_v; +)$  extending the group  $v$  in a natural way:

$$w := \{x \in k_v : \exists y \in v, |x| \leq y\}.$$

**Remark 1.1.1.** Given  $z \in R$  we may define a velocity  $\mathbb{N}z$  by setting:

$$\mathbb{N}z := \{x \in R : \exists N \in \mathbb{N}, |x| \leq Nz\}.$$

**Example 1.1.2.** Let  $k(0_+)$  be the field of real algebraic Puiseux series endowed with the order that makes the indeterminate  $T$  positive and smaller than any real number (see [BCR] example 1.1.2). Then, as in the above remark, the element  $T^k$  gives rise to a subgroup  $\mathbb{N}T^k$  which is constituted by all the series  $z$  having a valuation greater or equal to  $k$ . One could also consider the velocity  $v$  defined by the set of  $x$  satisfying  $|x| \leq NT^k$  for *any*  $N$  in  $\mathbb{N}$ . In the field of *ln-exp* definable germs of one variable functions (in a right-hand side neighborhood) one may consider the set of all the  $L^p$  integrable germs of series.

**Extension of functions.** On the other hand, as  $k_v$  is an elementary extension of  $R$ , it is well known that we may define  $X_v$ , the extension of  $X$  to  $k_v$ , by regarding the formula defining  $X$  in  $k_v^n$ . Every mapping  $\sigma : X \rightarrow Y$  may also be extended to a mapping  $\sigma_v : X_v \rightarrow Y_v$ .

**1.2.  $v$ -thin sets.** We give the definition of the  $v$ -thin sets which is required to introduce the vanishing homology.

**Definitions 1.2.1.** Let  $j \leq n$  be integers. A  $j$ -dimensional definable subset  $X$  of  $R^n$  is called  **$v$ -thin** if there exists  $z \in v$  such that, for any linear projection  $\pi : R^n \rightarrow R^j$ , no ball (in  $R^j$ ) of radius  $z$  entirely lies in  $\pi(X)$ .

For simplicity we say that  $X$  is  **$(j; v)$ -thin** if either  $X$  is  $v$ -thin or  $\dim X < j$ . A set which is not  $v$ -thin will be called  **$v$ -thick**.

Note that in the above definition it is actually enough to require that the property holds for a sufficiently generic projection  $\pi : R^n \rightarrow R^j$ . As we said in the introduction, roughly speaking,  $\mathbb{N}T^2$ -thin sets of  $k(0_+)^n$  are the generic fibers of one parameter families whose fibers “collapse onto a lower dimensional subset at rate at least  $t^2$ ” (if  $t$  is the parameter

of the family). Also, by convention  $R^0 = \{0\}$  so that a 0-dimensional subset is never  $v$ -thin. This is natural in the sense that a family of points never collapses onto a lower dimensional subset.

**Basic properties of  $(j; v)$ -thin sets.** (1) If a definable subset  $A \subset X$  is  $(j; v)$ -thin and if  $h : X \rightarrow Y$  is a definable Lipschitz map then  $h(A)$  is  $(j; v)$ -thin.

(2) Given  $j$ ,  $\cup_{i=1}^p X_i$  is  $(j; v)$ -thin iff  $X_i$  is  $(j; v)$ -thin for any  $i = 1, \dots, p$ .

**1.3. Definition of the vanishing homology.** Given a definable set  $X$  let  $C_j^v(X)$  be the  $G$ -submodule of  $C_j(X)$  generated by all the singular chains  $c$  such that  $|c|$  is  $(j; v)$ -thin and  $|\partial c|$  is  $(j; v)$ -thin as well. We endow this complex with the usual boundary operator and denote by  $Z_j^v(X)$  the cycles of  $C_j^v(X)$ .

A chain  $\sigma \in C_j^v(X)$  is said  **$v$ -admissible**. We denote by  $H_j^v(X)$  the resulting homology groups which we call the  **$v$ -vanishing homology groups**.

If  $v$  is  $\mathbb{N}z$ , for some  $z \in R$  (see Remark 1.1.1), then we will simply write  $C_j^z(X)$  and  $H_j^z(X)$  (rather than  $C_j^{\mathbb{N}z}$  and  $H_j^{\mathbb{N}z}$ ).

**Remark 1.3.1.** If  $X$  is  $v$ -thin and if  $j = \dim X$  then every  $j$ -chain is  $v$ -admissible. Moreover every  $(j + 1)$ -dimensional chain is admissible by definition. Hence the map  $H_j^v(X) \rightarrow H_j(X)$  induced by the inclusion of the chain complexes is an isomorphism. Note also that the map  $H_{j-1}^v(X) \rightarrow H_{j-1}(X)$  is a monomorphism.

Every Lipschitz map sends a  $(j; v)$ -thin set onto a  $(j; v)$ -thin set. Thus, every Lipschitz map  $f : X \rightarrow Y$ , where  $X$  and  $Y$  are two definable subsets, induces a sequence of mappings  $f_{j,v} : H_j^v(X) \rightarrow H_j^v(Y)$ . In consequence, the vanishing homology groups are preserved by definable bi-Lipschitz homeomorphisms.

As we said in the introduction this homology gives rise to a metric invariant for families (preserved by families of bi-Lipschitz homeomorphisms) by considering the generic fiber as described in the following example.

**Example 1.3.2.** With the notation of example 1.1.2, given an algebraic family  $X \subset \mathbb{R}^n \times \mathbb{R}$  defined by  $f_1 = \dots = f_p = 0$ , we set

$$X_{0+} := \{x \in k(0_+)^n : f_1(x; T) = \dots = f_p(x; T) = 0\}.$$

Hence,  $H_j^v(X_{0+})$  is a metric invariant of the family.

**1.4. The complex  $C_j^v(X; \mathcal{F})$ .** Given a finite family  $\mathcal{F}$ , of closed subsets of  $X$ , we write  $C_j(X; \mathcal{F})$  for the  $j$ -chains of  $\bigoplus_{F \in \mathcal{F}} C_j(F)$ . Similarly we set:

$$C_j^v(X; \mathcal{F}) := \bigoplus_{F \in \mathcal{F}} C_j^v(F)$$

and denote by  $H_j^v(X; \mathcal{F})$  the corresponding homology groups. By Remark 1.3.1, if  $\tau$  is a chain of  $Z_j^v(|\sigma|)$  whose class is  $\sigma$  in  $H_j(|\sigma|)$  then  $\tau = \sigma$  in  $H_j^v(|\sigma|)$  as well. Therefore, as  $H_j(|\sigma|; \mathcal{F}) = H_j(|\sigma|)$  we get:

$$(1.3) \quad H_j^v(X; \mathcal{F}) \simeq H_j^v(X).$$

**1.5. Strongly admissible chains.** It is difficult to construct homotopies between  $v$ -admissible chains. To overcome this difficulty we introduce strongly  $v$ -admissible chains.

**Definition 1.5.1.** We denote by  $T_j^q$  the set of all  $(x; \lambda) \in T_j \times R$  such that  $x + \lambda e_q$  belongs to  $T_j$ . A simplex  $\sigma : T_j \rightarrow R$  is **strongly  $v$ -admissible** if there exists  $q$  such that for any  $(x; \lambda) \in T_j^q$ :

$$(1.4) \quad (\sigma(x) - \sigma(x + \lambda e_q)) \in v.$$

A chain is strongly admissible if it is a combination of strongly admissible simplices. We denote by  $\widehat{C}_j^v(X)$  the chain complex generated by the strongly admissible chains  $\sigma$  for which  $\partial\sigma$  is strongly admissible, and by  $\widehat{Z}_j(X)$  the strongly admissible cycles. The resulting homology is denoted by  $\widehat{H}_j^v(X)$ . If  $\mathcal{F}$  is a family of closed subsets of  $X$ , we also define  $\widehat{C}_j^v(X; \mathcal{F})$ ,  $\widehat{Z}_j^v(X; \mathcal{F})$ , and  $\widehat{H}_j^v(X; \mathcal{F})$  in an analogous way (see section 1.4).

**Remark 1.5.2.** Let  $\sigma : T_j \rightarrow R^n$  be a strongly admissible simplex with  $j \leq n$ . Then by definition, there exists  $z \in v$  such that for any  $x \in T_j$ :

$$d(\sigma(x); \sigma(\partial T_j)) \leq z.$$

As  $\sigma(\partial T_j)$  is of dimension strictly inferior to  $j$  we see that the image of this set under a projection onto  $R^j$  contains no open ball in  $R^j$ . In other words, if  $\sigma$  and  $\partial\sigma$  are strongly admissible chains then  $\sigma$  is admissible. In consequence, a strongly admissible cycle is admissible.

## 2. RECTILINEARIZATIONS OF $v$ -THIN SETS.

**2.1. Regular directions.** We recall a result proved in [V1] which will be very useful to compute our vanishing homology. We start by the definition of a regular direction. We denote by  $X_{reg}$  the set of points  $x \in X$  at which  $X$  is a  $C^1$  manifold.

**Definition 2.1.1.** Let  $X$  be a definable set of  $R^n$ . An element  $\lambda$  of  $S^{n-1}$  is said **regular for  $X$**  if there exists a positive  $\alpha \in \mathbb{Q}$ :

$$d(\lambda; T_x X_{reg}) \geq \alpha,$$

for any  $x \in X_{reg}$ .

Not every definable set has a regular line. However, we have:

**Proposition 2.1.2.** [V1] *Let  $A$  be a definable subset of  $R^n$  of empty interior. Then there exists a definable bi-Lipschitz homeomorphism  $h : R^n \rightarrow R^n$  such that  $e_n$  is regular for  $h(A)$ .*

**Remark 2.1.3.** When  $e_n$  is regular for a set  $X$ , we may find finitely many Lipschitz definable functions, say  $\xi_i : R^{n-1} \rightarrow R$ ,  $i = 1, \dots, s$ , satisfying

$$(2.5) \quad \xi_1 \leq \dots \leq \xi_s,$$

and such that the set  $X$  is included in the union of their respective graphs.



**2.2. Cell decompositions.** In order to fix notations we recall the definition of the cells, which, as usual, are introduced inductively. All the definitions of this section deal with subsets of  $R^n$ , but since  $R$  stands for an arbitrary real closed field, we will use them for subsets of  $k_v^n$  as well.

**Definitions 2.2.1.** For  $n = 0$  a cell of  $R^n$  is  $\{0\}$ . A **cell**  $E$  of  $R^n$  is either the graph of a definable function  $\xi : E' \rightarrow R$ , where  $E'$  is a cell of  $R^{n-1}$  or a band of type:

$$(2.6) \quad \{x = (x'; x_n) \in E' \times R : \xi_1(x') < x_n < \xi_2(x')\},$$

where  $\xi_1, \xi_2 : E' \rightarrow R$  are two definable functions satisfying  $\xi_1 < \xi_2$  or  $\pm\infty$ . The cell  $E$  is **Lipschitz** if  $E'$  is Lipschitz and if  $\xi_1$  and  $\xi_2$  (or  $\xi$ ) are Lipschitz functions (and  $\{0\}$  is Lipschitz). A **closed cell** is the closure of a cell (which is obtained by replacing  $<$  by  $\leq$  in the definition).

Given  $z \in R$ , the Lipschitz cell  $E$  is  **$z$ -admissible** if

- (1)  $E'$  is  $z$ -admissible
- (2) If  $E$  is a band defined by two functions  $\xi_1$  and  $\xi_2$ , then either  $(\xi_2 - \xi_1)(x) \leq z$  for any  $x \in E'$ , or  $(\xi_2 - \xi_1)(x) \geq z$  for any  $x \in E'$ .

Set also that the cell  $\{0\}$  is  $z$ -admissible.

A cell  $E$  of dimension  $j$  is canonically homeomorphic to  $(0; 1)^j$ . The **barycentric subdivision** of  $E$  is the partition defined by the image by this homeomorphism of the barycentric subdivision of  $(0; 1)^j$ .

We shall need the following very easy lemma.

**Lemma 2.2.2.** *Let  $E'$  be a  $w$ -thick Lipschitz cell of  $k_v^{n-1}$  and let  $\xi_s : E' \rightarrow k_v$ ,  $s = 1, 2$ , be two Lipschitz functions such that  $\xi_1 < \xi_2$  and  $(\xi_1 - \xi_2)(x) \notin w$ , for any  $x \in E'$ . Then the band:*

$$E := \{(x; y) \in E' \times k_v : \xi_1(x) < y < \xi_2(x)\}$$

*is  $w$ -thick.*

*Proof.* We may assume that  $E'$  is open in  $k_v^{n-1}$  since we may find a bi-Lipschitz homeomorphism which carries  $E'$  onto an open cell. Then  $E$  is also an open and the cell  $E'$  contains a ball of radius  $z \notin w$ , say  $B(x_0; z)$ . Let  $t := \xi_2(x_0) - \xi_1(x_0)$ ; we have by assumption  $t \notin w$ . Taking  $t$  small enough we may assume  $B(x_0; t)$  entirely lies in  $E'$ . Let  $N$  be the Lipschitz constant of  $(\xi_1 - \xi_2)$  and note that:

$$\xi_2(x) - \xi_1(x) \geq \frac{t}{2},$$

for  $x \in B(x_0; \frac{t}{2N})$ . This implies that  $E$  contains a ball of radius  $\frac{t}{2N}$ . But, as  $t \notin w$  we have  $\frac{t}{2N} \notin w$ .  $\square$

**Definition 2.2.3.** The subset  $\{0\}$  is an  **$L$ -cell decomposition** of  $R^0$ . For  $n > 0$ , an  **$L$ -cell decomposition** of  $R^n$  is a cell decomposition of  $R^n$  satisfying:

- (i) The cells of  $R^{n-1}$  constitute an  $L$ -cell decomposition of  $R^{n-1}$
- (ii) There exist finitely many Lipschitz functions  $\xi_1, \dots, \xi_s : R^{n-1} \rightarrow R$  satisfying (2.5) such that the union of all the cells which are graphs of a function on a subset of  $R^{n-1}$ , is the union of the graphs of the  $\xi_i$ 's.



An  $L$ -cell decomposition is said **compatible with** finitely many given definable subsets  $X_1, \dots, X_m$  if these subsets are union of cells. It is said  **$z$ -admissible** if every cell is  $z$ -admissible. Taking the barycentric subdivision of every cell, we get a **barycentric subdivision** of an  $L$ -cell decomposition.

We are going to show that, we may find a  $u$ -admissible  $L$ -cell decomposition which is compatible with some given  $\mathcal{L}_R(u)$ -definable subsets of  $k_v^n$ . This will be helpful to prove that the homology groups are finitely generated, since we will show that only the  $\mathbb{N}u$ -thin cells are relevant to compute the homology groups. The following proposition deals with subsets of  $k_v$  since we will apply it to  $k_v$  but of course the proof goes over an arbitrary model of the theory.

**Proposition 2.2.4.** *Let  $X_1, \dots, X_m$  be  $\mathcal{L}_R(u)$ -definable subsets of  $k_v^n$ . There exists a  $\mathcal{L}_R(u)$  definable bi-Lipschitz homeomorphism  $h : k_v^n \rightarrow k_v^n$  such that we can find a  $u$ -admissible  $L$ -cell decomposition of  $k_v^n$  compatible with  $h(X_1), \dots, h(X_m)$ .*

*Proof.* For  $n = 0$  there is nothing to prove. Assume  $n > 1$  and apply Proposition 2.1.2 to  $\cup_{j=1}^m \partial X_j$  (where  $\partial$  denotes the topological boundary). Then (see Remark 2.1.3) there exist finitely many definable Lipschitz functions  $\xi_i$ ,  $i = 1, \dots, s$  satisfying (2.5). Consider a cell decomposition of  $k_v^n$  compatible with  $X_1, \dots, X_m$ , all the graphs of the  $\xi_i$ 's, as well as all the sets

$$\{x \in k_v^{n-1} : \xi_{i+1}(x) - \xi_i(x) = u\}.$$

Now apply the induction hypothesis to all the cells of this decomposition which lie in  $k_v^{n-1}$  to get a cell decomposition  $\mathcal{E}$  of  $k_v^{n-1}$ . Then set  $\xi_0 := -\infty$ ,  $\xi_{s+1} := \infty$ , and consider the cell decomposition of  $k_v^n$  constituted by the graphs of the restrictions of the functions  $\xi_i$ 's to an element of  $\mathcal{E}$  on the one hand, and all the subsets of type:

$$\{(x; x_n) \in E \times k_v : \xi_i(x) < x_n < \xi_{i+1}(x)\},$$

where  $E \in \mathcal{E}$ , on the other hand. The required properties hold.  $\square$

**2.3. Rectilinearization of  $v$ -thin sets.** We introduce the notion of rectilinearization. This is a mapping which transforms a set into a union of coordinate hyperplanes and which induces an isomorphism in homology (the usual one). Admissible rectilinearizations will be very helpful to construct strongly admissible chains (see section 1.5). We are going to show that we can always find a  $v$ -admissible rectilinearization compatible with a given family of  $v$ -thin sets.

**Definitions 2.3.1.** A **hyperplane complex** is a subset  $W$  of  $R^n$ , which is a union of finitely many coordinate hyperplanes of type  $x_j = s$  where, for each hyperplane,  $s$  is an integer. There is a canonical cell decomposition of  $R^n$  compatible with  $W$ . We refer to the cells (resp. closure of the cells) as the **cells of  $W$**  (resp. **closed cells of  $W$** ).

Let  $X_1, \dots, X_m$  be definable subsets. A **rectilinearization** of  $X_1, \dots, X_m$  is a mapping  $h : R^n \rightarrow R^n$ , such that the  $h^{-1}(X_i)$ 's are union of cells of  $W$  and such that for any  $i = 1, \dots, m$  the mapping  $h_i : h^{-1}(X_i) \rightarrow X_i$  induces an isomorphism in homology (the usual one).

If  $X_1, \dots, X_m$  are  $v$ -thin, a rectilinearization of  $X_1, \dots, X_m$  is  **$v$ -admissible** if for each cell  $\sigma$  of  $W$  included in  $h^{-1}(X_i)$  there exists an integer  $q$  with  $e_q$  tangent to  $\sigma$  for which

$$(2.7) \quad (h(x) - h(x + \lambda e_q)) \in v$$

for any  $x \in \sigma$  and  $\lambda \in R$  such that  $x + \lambda e_q \in \sigma$ .

**Remark 2.3.2.** After a barycentric subdivision of  $h^{-1}(X_i)$ , we get a simplicial complex  $K_i$  and a map  $h_i : K_i \rightarrow X_i$  which induces an isomorphism in homology. Note that, thanks to (2.7) each simplicial chain gives rise (identifying each  $j$ -simplex to  $T_j$  in a linear way) to a strongly admissible chain (see Definition 1.5.1). Moreover, as  $h$  induces an isomorphism in homology, this identification defines an isomorphism in homology  $H_j(K_i) \rightarrow H_j(X_i)$ .

**Proposition 2.3.3.** *Let  $X_1, \dots, X_m$  be closed definable  $v$ -thin subsets of  $R^n$ . Then there exists a  $v$ -admissible rectilinearization of  $X_1, \dots, X_m$ .*

*Proof.* We start by proving the following statements (**H<sub>n</sub>**) by induction on  $n$ .

(**H<sub>n</sub>**). Let  $\mathcal{E}$  be a  $u$ -admissible  $L$ -cell decomposition of  $k_v^n$  and let  $Y_1, \dots, Y_r$  denote the  $w$ -thin closed cells. Then there exists a  $\mathbb{N}u$ -admissible rectilinearization  $h : k_v^n \rightarrow k_v^n$  of  $Y_1, \dots, Y_r$  such that, for every  $E$  in  $\mathcal{E}$ ,  $h^{-1}(cl(E))$  is a union of closed cells of  $W$  and there exists a strong deformation retract  $r_E : h^{-1}(cl(E)) \rightarrow C_E$ , where  $C_E$  is a closed cell of  $W$ .

Note that it follows from the existence of this deformation retract that  $h$  induces an isomorphism in homology above any union of closed cells of  $\mathcal{E}$ . Actually, the existence of  $r_{Y_i}$  implies

$$H_j(h^{-1}(Y_i)) \simeq H_j(C_{Y_i}) \simeq H_j(Y_i),$$

and the map  $h|_{h^{-1}(Y_i)} : h^{-1}(Y_i) \rightarrow Y_i$  induces an isomorphism in homology. Therefore, thanks to the Mayer-Vietoris property and to the 5-Lemma, we see that for any subset  $X$  constituted by the union of finitely many closed cells the map  $h|_{h^{-1}(X)} : h^{-1}(X) \rightarrow X$  induces an isomorphism in homology.

Note that nothing is to be proved for  $n = 0$  and assume (**H<sub>n-1</sub>**). Apply the induction hypothesis to the family constituted by the closure of the cells of  $\mathcal{E}$  in  $k_v^{n-1}$  which are  $w$ -thin to get a rectilinearization  $h : k_v^{n-1} \rightarrow k_v^{n-1}$  and a hyperplane complex  $W$ .

Note that by definition, the cells of  $\mathcal{E}$  on which the restriction of  $\pi_n$  is one-to-one are included in the union of finitely many graphs of definable Lipschitz functions  $\xi_1, \dots, \xi_s : k_v^{n-1} \rightarrow k_v$  satisfying (2.5).

We obtain a hyperplane complex  $\widetilde{W}$  by taking the inverse image of  $W$  by  $\pi_n$ , and by adding the hyperplanes defined by  $x_n = i$ ,  $i = 1, \dots, s$ .

Define now the desired mapping  $\widetilde{h}$  as follows:

$$\widetilde{h}(x; i + t) = (h(x); (1 - t)\xi_i(h(x)) + t\xi_{i+1}(h(x)))$$

for  $1 \leq i < s$  integer,  $x \in k_v^n$  and  $t \in [0; 1)$ . Define also:

$$\widetilde{h}(x; 1 - t) = (h(x); \xi_1(h(x)) - t)$$

and

$$\widetilde{h}(x; s + t) = (h(x); \xi_s(h(x)) + t)$$

for  $t \in [0; \infty)$ . This defines a mapping  $\tilde{h} : k_v^n \rightarrow k_v^n$ . We are going to check that

$$(2.8) \quad |\tilde{h}(x) - \tilde{h}(x + \lambda e_n)| \leq u$$

when  $x$  and  $(x + \lambda e_n)$  belong to the same cell.

Let  $\sigma$  be a cell of  $\widetilde{W}$  which is mapped into  $\cup_{i=1}^r Y_i$ . If  $\pi_n(\sigma)$  is  $w$ -thin (2.8) follows from the induction hypothesis. Otherwise  $\tilde{h}(\sigma)$  must lie in the band delimited by the graphs of the restrictions of  $\xi_i$  and  $\xi_{i+1}$  for some  $i \in \{1, \dots, s-1\}$  as described in (2.6). If  $\tilde{h}(\pi_n(\sigma))$  fails to be  $w$ -thin then, thanks to Lemma 2.2.2 (recall that  $\tilde{h}(\sigma)$  is  $w$ -thin) and the  $u$ -admissibility of the cell decomposition, we necessarily have:

$$|\xi_i(x) - \xi_{i+1}(x)| \leq u,$$

for any  $x \in \pi_n(\sigma)$ . This, together with definition of  $\tilde{h}$ , implies that  $\tilde{h}$  satisfies (2.8) and yields that  $\tilde{h}$  is  $Nu$ -admissible. It remains to find the retraction  $r_E$  for each cell  $E$ .

Fix  $E \in \mathcal{E}$  and observe that it follows from the definition of  $\tilde{h}$  and the induction hypothesis that  $\tilde{h}^{-1}(cl(E))$  is a union of cells of  $\widetilde{W}$ . If  $E$  is the graph of a function  $\xi : E' \rightarrow k_v$  (where  $E' := \pi_n(E)$ ), then the result directly follows from the induction hypothesis. Otherwise, since  $\mathcal{E}$  is an  $L$ -cell decomposition, the cell  $E$  lies in the band delimited by the graphs of two consecutive functions, say  $\xi_i$  and  $\xi_{i+1}$ . Let

$$\Gamma_i := \{(x; x_n) \in k_v^{n-1} \times k_v : i \leq x_n \leq i+1\}.$$

We first define first a retract:

$$r'_E : \tilde{h}^{-1}(cl(E)) \times [0; \frac{1}{2}]_{k_v} \rightarrow \Gamma_i \cap \tilde{h}^{-1}(cl(E)),$$

by setting for  $x_n \geq i+1$ :

$$r'_E(x; x_n; t) := (x; 2tx_n + (1-2t)(i+1)),$$

and for  $x_n \leq i$ :

$$r'_E(x; x_n; t) := (x; 2tx_n + (1-2t)i),$$

and of course  $r'_E(x; x_n; t) := (x; x_n)$  when  $i \leq x_n \leq i+1$ .

Note that it follows from the definition of  $\tilde{h}$  that if  $(x; x_n)$  belongs to  $\tilde{h}^{-1}(cl(E))$  then for any  $i+1 \leq x'_n \leq x_n$  and any  $x_n \leq x'_n \leq i$ :

$$\tilde{h}(x; x'_n) = \tilde{h}(x; x_n).$$

This implies that  $r'_E$  preserves  $\tilde{h}^{-1}(cl(E))$ .

On the other hand, thanks to the induction hypothesis, there exists a retract  $r_{E'} : h^{-1}(cl(E')) \times [0; 1]_{k_v} \rightarrow C_{E'}$ . Let us extend this  $r_{E'}$  into a retract:

$$r''_{E'} : \pi_n^{-1}(h^{-1}(cl(E'))) \times [\frac{1}{2}; 1]_{k_v} \rightarrow \pi_n^{-1}(C_{E'})$$

by

$$r'_E(x; x_n; t) := (r_{E'}(x; 2t-1); x_n).$$

Clearly, there exists a unique cell  $C_E$  of  $\widetilde{W}$  which is included in  $\Gamma_i$  and which projects on  $C_{E'}$ . Now, these retracts give rise to a retract

$$\tilde{r}_E : \tilde{h}^{-1}(cl(E)) \times [0; 1]_{k_v} \rightarrow C_E$$

defined by  $\tilde{r}_E(x; t) := r'_E(x; t)$  if  $t \leq \frac{1}{2}$  and

$$\tilde{r}_E(x; t) := r''_E(r'_E(x; \frac{1}{2}); t)$$

if  $t \geq \frac{1}{2}$ . This yields  $(\mathbf{H}_n)$ .

We return to the proof of the proposition. Apply Proposition 2.2.4 to  $X_{1,v}, \dots, X_{m,v}$ . This provides a bi-Lipschitz homeomorphism  $g : k_v^n \rightarrow k_v^n$  such that we can find a  $u$ -admissible  $L$ -cell decomposition of  $k_v^n$  compatible with  $g(X_{1,v}), \dots, g(X_{m,v})$ . Note that, as the  $g(X_{i,v})$ 's are  $w$ -thin, each of them is the union of some  $w$ -thin cells. Then by  $(\mathbf{H}_n)$ , there exists a  $Nu$ -admissible rectilinearization of these cells  $h : k_v^n \rightarrow k_v^n$ .

Composing with  $g$ , the mapping  $h$  gives rise to a  $Nu$ -admissible rectilinearization  $f$  of  $X_{1,v}, \dots, X_{m,v}$ . As the  $X_{i,v}$  are extensions, there exist two families of rectilinearizations  $f_z$  and  $h_z$  for  $z \in [a; b]$  with  $a < u < b$  and  $a, b \in R$ . Let us check that these rectilinearizations are  $v$ -admissible for  $z \in v$  large enough.

Note that each  $X_i$  is the union of the images by  $h_z$  of finitely many cells of  $W$ . Furthermore, as (2.8) is a first order formula we get that  $h_z$  satisfies on any given cell in the inverse image of the  $X_i$ 's:

$$|h_z(x) - h_z(x + \lambda e_n)| \leq z,$$

when  $x$  and  $(x + \lambda e_n)$  belong to this given cell.

This implies that  $f$  satisfies (since  $g$  is bi-Lipschitz):

$$|f_z(x) - f_z(x + \lambda e_n)| \leq Nz,$$

for some  $N \in \mathbb{N}$  and any  $z \in v$  large enough on any cell mapped into one of the  $X_i$ 's. Thus, (2.7) holds and  $f_z$  is  $w$ -admissible.  $\square$

**Remark 2.3.4.** Actually, working a little more, we could have proved that the constructed rectilinearization induces an isomorphism in homology above any subset  $A$  of  $R^n$ . Namely, in the above proof, given a subset  $A$  of  $R^n$ , the induced mapping  $\tilde{h} : \tilde{h}^{-1}(A) \rightarrow A$  induces an isomorphism in homology.

Observe also that the constructed mapping is a homeomorphism above a dense definable subset. If we take an algebraic hypersurface, the situation is fairly similar to the one which occurs with resolution of singularities in the sense that the inverse image of the set above which the map is not one-to-one (the “exceptional divisor”) is constituted by finitely many coordinate hyperplanes normal to the hyperplanes lying above our given set. We could also have a more precise description of how the mapping  $h$  modifies the distances (like in [V1]). More precisely, it is possible to see that on each cell, we have

$$|h(x) - h(x')| \sim \sum_{i=1}^n \varphi_i(x) |x_i - x'_i|$$

where  $\varphi_i$  is a sum of product of powers of distances to cells of  $W$ . If we compare this result with Theorem 5.1.3 of [V1], we see that now the contractions (see [V1]) are expressed in the canonical basis. The inconvenient is that the map  $h$  is not a homeomorphism (contrarily as in [V1]), but since it induces an isomorphism between the homology groups, it will be enough for the purpose of the present paper.

## 3. THE VANISHING HOMOLOGY GROUPS ARE FINITELY GENERATED

**3.1. Some preliminary lemmas.** Every mapping  $\sigma : T_j \rightarrow X$  may be extended to a mapping  $\sigma_v : T_j(k_v) \rightarrow X_v$  (see subsection 1.1). Let  $\Delta_j^{ext}(X_v)$  be the submodule of  $C_j^w(X_v)$  generated by the simplices which are extensions of an element of  $C_j^v(X)$ . Clearly, for each  $j$  the mapping:

$$ext : C_j^v(X) \rightarrow \Delta_j^{ext}(X_v),$$

which assigns to every chain  $\sigma$  the chain  $\sigma_v$ , induces an isomorphism in homology.

The following Lemma says that the vanishing homology groups for the velocities  $\mathbb{N}u$  and  $w$  coincide with the homology groups of  $\Delta_j^{ext}$  when the considered set is  $\mathcal{L}_R$ -definable.

**Lemma 3.1.1.** *Let  $X$  be a definable subset of  $R^n$ . Then the maps induced by the inclusions  $H_j(\Delta_j^{ext}(X_v)) \rightarrow H_j^u(X_v)$  and  $H_j(\Delta_j^{ext}(X_v)) \rightarrow H_j^w(X_v)$  are isomorphisms for any  $j$ .*

*Proof.* We do the proof for  $u$ . To get the proof for  $w$ , just replace  $u$  by  $w$ . We first check that this map is onto. Let  $\sigma = \sum_{i \in I} g_i c_i \in Z_j^u(X_v)$ . By definition of  $u$  there exist finitely many  $\mathcal{L}_R$ -definable mappings, say  $\tau_i : T_j(k_v) \times [a; u]_{k_v} \rightarrow X_v$ , with  $a \in v$  such that  $c_i(x) = \tau_i(x; u)$  for any  $x \in T_j(k_v)$ . Define  $\theta_i(x) := \tau_i(x; a)$  and  $\theta := \sum_{i \in I} g_i \theta_i \in C_j^{ext}(X_v)$ . Observe that  $\tau_i$  gives rise to a  $\mathbb{N}u$ -admissible  $(j+1)$ -chain (after a subdivision of  $T_j(k_v) \times [a; u]$ ). Moreover, as the property of admissibility may be expressed by a formula with parameters in  $R$  and with  $u$ , we know that the obtained chain is  $\mathbb{N}u$ -admissible if  $a$  is chosen large enough. Set  $\tau := \sum_{i \in I} g_i \tau_i \in C_{j+1}^u(X_v)$  and note that since  $\tau_i(x; u) = c_i(x)$  and  $\tau_i(x; a) = \theta_i(x)$  we clearly have  $\partial\tau = \sigma - \theta$ . As  $\theta$  belongs to  $C_j^{ext}(X_v)$ , this implies that the inclusion  $C_j^{ext}(X_v) \rightarrow C_j^u(X_v)$  induces a surjection in homology.

We now check that this map is injective by applying a similar argument. Let  $\alpha \in C_j^{ext}(X_v)$  with  $\alpha = \partial\sigma$  where  $\sigma$  belongs to  $C_{j+1}^u(X_v)$ . The chain  $\sigma$  induces chains  $\tau \in C_{j+2}^u(X_v)$  and  $\theta \in C_{j+1}^{ext}(X_v)$  such that  $\partial\tau = \sigma - \theta$  in the same way as in the previous paragraph. But this implies  $\partial\theta = \alpha$  which means that  $\alpha \in \partial C_{j+1}^{ext}(X_v)$ , as required.  $\square$

Given a definable family  $Y$  of  $R^n \times R$  and  $t \in R$ , we denote by  $Y_t$  the **fiber at  $t$** :

$$\{x \in R^n : (x; t) \in Y\}.$$

We also define the **restriction of the family** to  $[a; b]$  as follows:

$$Y_{[a; b]} := \{(x; t) \in Y : a \leq t \leq b\}.$$

**Lemma 3.1.2.** *Let  $Y$  be a  $\mathcal{L}_R(u)$ -definable family of  $k_v^n \times k_v$  such that  $Y_u$  is a  $\mathbb{N}u$ -thin subset of  $k_v^n$  and let  $j = \dim Y_u$ . Then there exists  $z$  in  $v$  such that for any  $t \in v$  greater than  $z$  the map induced by inclusion:*

$$H_k^w(Y_t) \rightarrow H_k^u(Y_{[z; u]}),$$

*is an isomorphism for  $k = j$  and is one-to-one for  $k = j - 1$ .*

*Proof.* As  $Y$  is  $\mathcal{L}_R(u)$ -definable and  $\mathbb{N}u$ -thin there exists  $z$  in  $v$  such that for any  $t$  in  $v$  greater than  $z$ ,  $Y_t$  is  $w$ -thin. Thanks to Remark 1.3.1, this implies that the natural mapping  $H_j^w(Y_t) \rightarrow H_j(Y_t)$  is an isomorphism.

Furthermore, since the family  $Y$  is topologically trivial if the interval  $[z; u]$  is chosen small, the inclusion  $H_j(Y_t) \rightarrow H_j(Y_{[z; u]})$  induces an isomorphism in homology as well.

We have the following commutative diagram for  $t \in v$  greater than  $z$ :

$$\begin{array}{ccc} H_j^w(Y_t) & \xrightarrow{1} & H_j(Y_t) \\ \downarrow 3 & & \downarrow 2 \\ H_j^u(Y_{[z; u]}) & \xrightarrow{4} & H_j(Y_{[z; u]}) \end{array}$$

By the above, the arrows 1 and 2 are isomorphisms. Moreover as  $Y_u$  is  $\mathbb{N}u$ -thin the family  $Y_{[z; u]}$  is  $\mathbb{N}u$ -thin. Thus, the arrow 4 is an monomorphism (see the last sentence of Remark 1.3.1). This implies that the arrow 3 is an isomorphism and establishes the theorem in the case  $k = j$ .

Now, in the case where  $k = j - 1$  we can write the same diagram for  $H_{j-1}$ . The arrows 1 and 2 (of the obtained diagram) are still one-to-one (again thanks to Remark 1.3.1 and the topological triviality of  $Y_{[z; u]}$ ), so that the arrow 3 is clearly one-to-one.  $\square$

The following lemma is a consequence of existence of  $v$ -admissible rectilinearizations.

**Lemma 3.1.3.** *Given  $X \subset k_v^n \mathcal{L}_R(u)$ -definable and  $\mathcal{F}$  finite family of closed  $\mathcal{L}_R(u)$ -definable subsets of  $X$ , the map  $\widehat{H}_j^w(X; \mathcal{F}) \rightarrow H_j^w(X)$ , induced by the inclusion, is onto.*

*Proof.* Let  $\sigma \in C_j^w(X)$ . If the support of  $\sigma$  is of dimension  $< j$  then the class of  $\sigma$  is 0 in  $H_j^w(X)$ . Thus, we may assume that  $\dim|\sigma| = j$ .

Let  $h : k_v^n \rightarrow k_v^n$  be a  $w$ -admissible rectilinearization of  $|\sigma|$  and of all the elements of  $\mathcal{F}$ . There exists a simplicial chain  $\tau$  (see Remark 2.3.2), which is strongly  $w$ -admissible since  $h$  is  $w$ -admissible, such that  $\sigma = \tau$  in  $H_j(|\sigma|) = H_j^w(|\sigma|)$  (see Remark 1.3.1). But this means that the class of  $\tau$  is that of  $\sigma$  also in  $H_j^w(X)$ . This yields that the inclusion induces an onto map in homology.  $\square$

It is unclear for the author whether the inclusion of the above lemma is one-to-one. Actually, it is even unclear whether  $\widehat{H}_j^v(X)$  is finitely generated.

**3.2. The main result.** It is very hard to construct homotopies which are Lipschitz mappings. To compute the homology, we actually just need to find a homotopy that carries a chain  $\sigma$  to the cells of a given cell decomposition, and which preserves the  $v$ -admissibility of the chain  $\sigma$ . We prove something even weaker: given a strongly  $w$ -admissible chain, we may construct a homotopy which carries the chain  $\sigma$  to a strongly  $\mathbb{N}u$ -admissible chain of the cells of dimension  $j$ . This is enough since we have seen that we had isomorphisms between the theories defined by  $w$  and  $\mathbb{N}u$ . This technical step is performed in the following proposition.

**Proposition 3.2.1.** *Let  $X$  be a closed  $\mathcal{L}_R(u)$ -definable subset of  $k_v^n$  and let  $\mathcal{E}$  be a  $u$ -admissible  $L$ -cell decomposition compatible with  $X$ . Let  $\mathcal{F}$  be the family constituted by the*

closed cells of  $\mathcal{E}$  and let  $Y_j$  be the union of the closures of the  $(\mathbb{N}u; j)$ -thin elements of the barycentric subdivision of  $\mathcal{E}$ . Then, there exists a map

$$\varphi : \widehat{C}_j^w(X; \mathcal{F}) \rightarrow \widehat{C}_j^u(Y_j)$$

such that:

- (i)  $\varphi \partial - \partial \varphi = 0$
- (ii) For any  $\sigma \in \widehat{Z}_j^w(X; \mathcal{F})$  we have:  $\varphi_\sigma = \sigma$ , in  $H_j^u(X)$ ,
- (iii) If  $Y$  is the union of some elements of  $\mathcal{F}$ , then for any  $\sigma \in \widehat{Z}_j^w(X; \mathcal{F})$  with  $|\sigma| \subset Y$  we have:  $\varphi_\sigma = \sigma$  in  $H_j^u(Y)$ .

*Proof.* We are going to prove the following statements:

**Claim.** Given  $\sigma \in C_j(X; \mathcal{F})$ , there exists a definable homotopy

$$h_\sigma : T_j(k_v) \times [0; 1]_{k_v} \rightarrow X,$$

such that:

- (1) For each  $x$  the path  $t \mapsto h_\sigma(x; t)$  stays in the same closed cell,
- (2) For each  $t$  the map  $x \mapsto h_\sigma(x; t)$  is a strongly  $\mathbb{N}u$ -admissible simplex if  $\sigma$  is a strongly  $w$ -admissible simplex,
- (3) If  $\sigma$  is strongly  $w$ -admissible, the support of the simplex  $\varphi_\sigma : T_j(k_v) \rightarrow X$  defined by  $\varphi_\sigma(x) = h_\sigma(x; 1)$  entirely lies in  $Y_j$
- (4) We have

$$\partial h_*(\sigma) - h_*(\partial \sigma) = \varphi_\sigma - \sigma$$

for any  $\sigma \in C_j(X; \mathcal{F})$  where (as usual)  $h_* : C_j(X; \mathcal{F}) \rightarrow C_{j+1}(X; \mathcal{F})$  is the mapping induced by  $h$  on the chain complexes.

Note that  $\varphi$  is defined by (3). Observe that (4) implies (i), together with (2) implies (ii), and together with (1) yields (iii).

We prove that it is possible to construct such a homotopy by induction on  $n$  (the dimension of the ambient space). Let  $\mathcal{E}'$  be the cell decomposition of  $k_v^{n-1}$  constituted by all the cells of  $\mathcal{E}$  lying in  $k_v^{n-1}$ . Let  $\sigma$  in  $C_j(X; \mathcal{F})$  and write  $\sigma := (\tilde{\sigma}; \sigma_n) \in k_v^{n-1} \times k_v$ . Apply the induction hypothesis to  $\tilde{\sigma}$  and  $\mathcal{E}'$  to get a homotopy  $h_{\tilde{\sigma}} : T_j(k_v) \times [0; 1]_{k_v} \rightarrow k_v^{n-1}$ .

By definition, the union of the cells of  $\mathcal{E}$  on which  $\pi_n$  is one-to-one is given by the graphs of finitely many Lipschitz functions  $\xi_1 \leq \dots \leq \xi_s$ . Note that we may retract the cells above (resp. below) the graph of  $\xi_s$  (resp.  $\xi_1$ ) onto the graph of  $\xi_s$  (resp.  $\xi_1$ ) so that we may assume that  $X$  entirely lies between these two graphs.

By compatibility with  $\mathcal{F}$  we know that the support of  $\sigma$  entirely lies in one single cell  $E \in \mathcal{E}$  which is either the graph of a Lipschitz function  $\xi$  or a band which is delimited by the graph of the restriction to  $E' := \pi_n(E)$  of two consecutive functions  $\xi_i$  and  $\xi_{i+1}$ , with  $\xi_i < \xi_{i+1}$  on  $E'$ . In the latter case, we may define a function  $\nu_\sigma : T_j(k_v) \rightarrow [0; 1]_{k_v}$  by setting for  $x \in T_j(k_v)$

$$\nu_\sigma(x) := \frac{\sigma_n(x) - \xi_i(\tilde{\sigma}(x))}{\xi_{i+1}(\tilde{\sigma}(x)) - \xi_i(\tilde{\sigma}(x))}.$$

To deal with both cases simultaneously it is convenient to set  $\nu_\sigma(x) \equiv 0$  and  $\xi_i = \xi_{i+1} = \xi$ , if the cell is described by the graph of a single function  $\xi$ . To define  $h_\sigma$  we first define



a function  $s_\sigma : T_j(k_v) \rightarrow [0; 1]_{k_v}$ . We set:

$$\begin{aligned} s_\sigma(e_i) &= 0 & \text{if } \sigma_n(e_i) - \xi_i(\tilde{\sigma}(e_i)) \in w & \text{ and } \xi_{i+1}(\tilde{\sigma}(e_i)) - \sigma_n(e_i) \neq 0 \\ \text{and } s_\sigma(e_i) &= 1 & \text{otherwise.} \end{aligned}$$

Then we extend  $s_\sigma$  over  $T_j(k_v)$  linearly.

Now we can set for  $(x; t) \in T_j(k_v) \times [0; \frac{1}{2}]_{k_v}$ :

$$\theta(x; t) = 2ts_\sigma(x) + (2t - 1)\nu_\sigma(x).$$

Set for simplicity:  $\xi' = \xi_{i+1} - \xi_i$  and, for  $x = (\tilde{x}; x_n) \in k_v^{n-1} \times k_v$  and  $t \in [0; 1]_{k_v}$ , let:

$$\begin{aligned} h_\sigma(x; t) &:= (\tilde{\sigma}(x); \xi_i(\tilde{\sigma}(x)) + \theta(x; t)\xi'(\tilde{\sigma}(x))) & \text{if } t \leq \frac{1}{2} \\ h_\sigma(x; t) &:= (h_{\tilde{\sigma}}(\tilde{x}; 2t - 1); \xi_i(h_{\tilde{\sigma}}(\tilde{x}; 2t - 1)) + s_\sigma(x)\xi'(h_{\tilde{\sigma}}(\tilde{x}; 2t - 1))) & \text{if } t \geq \frac{1}{2}. \end{aligned}$$

Note that as  $s_\sigma$  (resp.  $\nu_\sigma$ ) satisfies:

$$s_{\partial\sigma} = \partial s_\sigma$$

(resp.  $\nu_{\partial\sigma} = \partial\nu_\sigma$ ), we see that the map induced by  $h_\sigma$  is a chain homotopy. Moreover, it is clear from the definition of  $h_\sigma$  that the path  $t \mapsto h_\sigma(x; t)$  remains in the same closed cells. Therefore (1) and (4) hold.

To check (2), fix a strongly admissible simplex  $\sigma$ . We have to check that there exists  $q \in \{1, \dots, n\}$  such that:

$$(3.9) \quad (h_\sigma(x + \lambda e_q; t) - h_\sigma(x; t)) \in \text{Nu}$$

for any  $(x; \lambda) \in T_j^q(k_v)$  and any  $t$  in  $[0; 1]_{k_v}$ .

If  $\sigma$  is the graph of one single function  $\xi$  then the result is immediate for  $t \leq \frac{1}{2}$  and follows from the induction hypothesis for  $t \geq \frac{1}{2}$ .

By definition of strongly admissible simplices there exists a vector of the canonical basis, say  $e_q$ , such that:

$$(3.10) \quad (\sigma(x) - \sigma(x + \lambda e_q)) \in w,$$

for any  $(x; \lambda) \in T_j^q(k_v)$ . This implies that

$$(3.11) \quad (\sigma(0) - \sigma(e_q)) \in w.$$

We distinguish two cases:

First case:  $s_\sigma(0) = s_\sigma(e_q)$ . This implies that for any  $(x; \lambda) \in T_j^q(k_v)$  we have

$$s_\sigma(x) = s_\sigma(x + \lambda e_q),$$

and therefore

$$(3.12) \quad |\theta(x) - \theta(x + \lambda e_q)| \leq |\nu_\sigma(x) - \nu_\sigma(x + \lambda e_q)|.$$

Note that if  $\xi'(\tilde{\sigma}(x)) \in w$  then  $\xi'(\tilde{\sigma}(x + \lambda e_q)) \in w$ , which means that in this case (3.9) follows immediately from (3.10) for  $t \leq \frac{1}{2}$ . Otherwise  $\xi'(\tilde{\sigma}(x)) \notin w$  and then by (3.10):

$$(3.13) \quad \frac{1}{2}\xi'(\tilde{\sigma}(x)) \leq \xi'(\tilde{\sigma}(x + \lambda e_q)) \leq 2\xi'(\tilde{\sigma}(x)).$$

Recall that the functions  $\xi_i$  and  $\xi_{i+1}$  are both Lipschitz functions. Hence, if  $\sigma$  is strongly admissible, for  $t \leq \frac{1}{2}$  a straightforward computation shows that thanks to (3.12) and (3.13) we have for any  $(x; \lambda) \in T_j^q(k_v)$ :

$$(3.14) \quad (h_\sigma(x + \lambda e_q; t) - h_\sigma(x; t)) \in w.$$

For  $t \geq \frac{1}{2}$ , (3.9) still holds thanks to the induction hypothesis and the Lipschitzness of  $\xi_i$  and  $\xi_{i+1}$ .

Second case:  $s_\sigma(0) \neq s_\sigma(e_q)$ . In this case we observe that if  $s_\sigma(0)$  is 0 then

$$(\sigma_n(0) - \xi_i(\tilde{\sigma}(0))) \in w$$

which amounts to

$$d(\sigma(0); \Gamma_{\xi_i}) \in w,$$

(where  $\Gamma_{\xi_i}$  denotes the graph of  $\xi_i$ ). By (3.11), this implies that  $d(\sigma(e_q); \Gamma_{\xi_i})$  belongs to  $w$  and so

$$(\sigma_n(e_q) - \xi_i(\tilde{\sigma}(e_q))) \in w.$$

As  $s_\sigma(e_q)$  is necessarily equal to 1 we see that

$$\sigma_n(e_q) - \xi_{i+1}(\tilde{\sigma}(e_q)) = 0$$

so that

$$\xi'(\tilde{\sigma}(e_q)) \in w.$$

But, as the cell  $E$  is  $u$ -admissible this implies that for any  $x \in E'$ :

$$\xi'(x) \leq u.$$

This, together with the induction hypothesis, implies that  $h_\sigma$  satisfies (3.9). This completes the proof of (2).

It remains to prove (3). First observe that all the  $e_j$ 's are sent by  $\varphi_\sigma$  onto vertices of  $E$ . Note also that

$$\varphi_\sigma(x) = (\varphi_{\tilde{\sigma}}(x); \xi_i(\varphi_{\tilde{\sigma}}(x)) + s_\sigma(x)\xi'(\varphi_{\tilde{\sigma}}(x)))$$

and so, by the definition of the cells, the support of  $\varphi_\sigma$  lies in cells of dimension at most  $j$  of  $\mathcal{F}$ . Moreover we just checked that (3.9) holds in any case. This implies that  $\varphi_\sigma$  is strongly admissible and therefore its support must lie in  $Y_j$ . This completes the proof of the claim.  $\square$

We are now able to express the  $v$ -vanishing homology groups in terms of the (usual) homology groups of some  $v$ -thin subsets constituted by the  $v$ -thin cells of the barycentric subdivision of some  $L$ -cell decompositions.

**Theorem 3.2.2.** *For any  $X \subset R^n$  closed definable, there exist some definable subsets of  $X$ :*

$$X_0 \subset \cdots \subset X_{d+1} = X_d$$

*such that:*

$$H_j^v(X) \simeq \text{Im}(H_j(X_j) \rightarrow H_j(X_{j+1}))$$

*(where the arrow is induced by inclusion and  $\text{Im}$  stands for image).*

*Proof.* We start by defining inductively the subsets  $X_j$  's. Set  $X_0 = \emptyset$  and assume that  $X_0, \dots, X_{j-1}$  have already been defined. According to Proposition 2.2.4, up to a bi-Lipschitz homeomorphism, we can assume that we have a  $u$ -admissible  $L$ -cell decomposition compatible with  $X_v$  and  $X_{j-1,v}$ . Let  $\mathcal{E}_j$  be the barycentric subdivision of this cell decomposition and define  $\Theta_j$  as the union of all the  $(j; \mathbb{N}u)$ -thin cells. There exists a  $\mathcal{L}_R$ -definable family  $Y_j$  such that  $Y_{j,u} = \Theta_j$ . Now, thanks to Lemma 3.1.2, there exists  $z$  in  $v$ , such that for any  $t$  in  $v$  greater than  $z$ :

$$H_j^w(Y_{j,t}) \simeq H_j^u(Y_{j,u}).$$

Now define  $X_j$  as the subset of  $R^n$  defined by a  $\mathcal{L}_R$ -formula defining  $Y_{j,t}$  for some  $t \geq z$  in  $v$ . If  $t$  is chosen large enough,  $X_j$  is  $v$ -thin. As bi-Lipschitz homeomorphisms induce isomorphisms between the vanishing homology groups, we identify subsets with their image so that, for instance, we consider below the  $X_{j,v}$ 's and  $Y_{j,u}$  as subsets of  $X_v$ .

Consider the following diagram:

$$\text{Im}\{H_j(X_j) \rightarrow H_j(X_{j+1})\} \xleftarrow{a} \text{Im}\{H_j^v(X_j) \rightarrow H_j^v(X_{j+1})\} \xrightarrow{b} H_j^v(X),$$

where again  $a$  and  $b$  are induced by the inclusions of the corresponding chain complexes. We shall show that  $a$  and  $b$  are both isomorphisms.

$a$  is an isomorphism: We have the following commutative diagram:

$$\begin{array}{ccc} H_j^v(X_j) & \longrightarrow & H_j^v(X_{j+1}) \\ \downarrow & & \downarrow \\ H_j(X_j) & \longrightarrow & H_j(X_{j+1}) \end{array}$$

where all the maps are induced by inclusion. By Remark 1.3.1, the first vertical arrow is an isomorphism and the second is one-to-one. This proves that  $a$  is an isomorphism.

$b$  is onto: Note that it is enough to prove that the inclusion  $X_j \rightarrow X$  induces an onto map between the  $v$ -vanishing homology groups.

We have the following commutative diagram:

$$\begin{array}{ccccc} H_j^v(X_j) & \xrightarrow{\text{ext}} & H_j(\Delta^{\text{ext}}(X_{j,v})) & \longrightarrow & H_j^w(X_{j,v}) \\ \downarrow & & & & \downarrow \\ H_j^v(X) & \xrightarrow{\text{ext}} & H_{j+1}(\Delta^{\text{ext}}(X_v)) & \longrightarrow & H_j^w(X_v) \end{array} \quad \text{diag. 1.}$$

where the mapping  $\text{ext}$ , provided by extension of chains, is an isomorphism (see section 3.1).

By Lemma 3.1.1 the latter horizontal arrows are isomorphisms as well. Therefore, it is enough to prove that the map induced by inclusion  $H_j^w(X_{j,v}) \rightarrow H_j^w(X_v)$  (the last vertical arrow) is onto.

For  $t \geq z$  in  $v$ , let  $\alpha$  and  $\beta$  be the maps defined by inclusion:

$$H_j^w(Y_{j,t}) \xrightarrow{\alpha} H_j(Y_{j,[z;u]}) \xleftarrow{\beta} H_j^u(Y_{j,u}).$$

By Lemma 3.1.2,  $\alpha$  and  $\beta$  are isomorphisms so that  $\gamma := \beta^{-1}\alpha$  provides the following commutative diagram:

$$\begin{array}{ccc} H_j^w(Y_{j,t}) & \longrightarrow & H_j^w(X_v) \\ \gamma \downarrow & & \downarrow \\ H_j^u(Y_{j,u}) & \longrightarrow & H_j^u(X_v) \end{array}$$

By Lemma 3.1.1 the second vertical arrow is onto. Thus, it is enough to show that  $H_j^u(Y_{j,u}) \rightarrow H_j^u(X_v)$  is onto. By construction,  $Y_{j,u}$  is the union of all the  $(j; Nu)$ -thin closed cells of the barycentric subdivision of  $\mathcal{E}_j$ . Note that it is enough to consider a chain  $\sigma \in \widehat{Z}_j^w(X_v; \mathcal{F})$  where  $\mathcal{F}$  is the family constituted by all the closure of the cells of  $\mathcal{E}_j$  (since the inclusion  $\widehat{H}_j^w(X_v; \mathcal{F}) \rightarrow H_j^w(X_v)$  is onto, thanks to Lemmas 3.1.1 and 3.1.3). By (ii) of Proposition 3.2.1, there exists  $\varphi_\sigma \in C_j^u(Y_{j,u})$  such that  $\sigma = \varphi_\sigma$  in  $H_j^u(X_v)$ , as required.

$b$  is one-to-one: Note that as diag. 1. holds for  $X_{j+1}$  as well (and the horizontal arrows are isomorphisms as well), it is enough to show that the map induced by inclusion

$$b' : \text{Im}(H_j^w(X_{j,v}) \rightarrow H_j^w(X_{j+1,v})) \rightarrow H_j^w(X_v)$$

is one-to-one. Recall that by definition  $X_{j+1,v}$  is  $Y_{j+1,t}$ , for some  $t$  and consider the following commutative diagram:

$$\begin{array}{ccc} H_j^w(X_{j,v}) & \longrightarrow & H_j^w(Y_{j+1,t}) \\ \downarrow & & \downarrow \nu_t \\ H_j^u(Y_{j+1,u}) & \xrightarrow{\nu_u} & H_j^u(Y_{j+1,[z;u]}) \end{array}$$

where again  $\nu_u$  and  $\nu_t$  are induced by the respective inclusions. By Lemma 3.1.2 these maps are one-to-one.

This implies that we have the following commutative diagram:

$$\begin{array}{ccc} \text{Im}(H_j^w(X_{j,v}) \rightarrow H_j^w(Y_{j+1,t})) & \xrightarrow{b'} & H_j^w(X_v) \\ \mu \downarrow & & \downarrow \\ \text{Im}(H_j^w(X_{j,v}) \rightarrow H_j^u(Y_{j+1,u})) & \xrightarrow{b''} & H_j^u(X_v) \end{array}$$

where all the horizontal arrows are induced by the corresponding inclusions and  $\mu$  is induced by the restriction of  $\nu_u^{-1}\nu_t$ . Since  $\mu$  is one-to-one, it is enough to show that  $b''$  is one-to-one.

To check that  $b''$  is one-to-one, take  $\sigma$  in  $Z_j^w(X_{j,v})$  which bounds a chain of  $C_{j+1}^u(X_v)$ . As the inclusion  $H_j^w(X_v) \rightarrow H_j^u(X_v)$  is an isomorphism, there exists  $\tau$  in  $C_{j+1}^w(X_v)$  such that  $\sigma = \partial\tau$ . Consider a  $w$ -admissible rectilinearization of  $|\tau|$ ,  $|\sigma|$  and  $\mathcal{F}$  where  $\mathcal{F}$  is the family constituted by the closure the cells of the barycentric subdivision of  $\mathcal{E}_{j+1}$ . The chain  $\sigma$  is equal in  $H_j(X_{j,v}) \simeq H_j^w(X_{j,v})$  (see Remark 2.3.2) to a simplicial chain  $\sigma'$  which is strongly  $w$ -admissible and compatible with  $\mathcal{F}$ . The class of the chain  $\sigma$  is zero in  $H_j(|\tau|)$  and therefore  $\sigma'$  bounds a simplicial chain  $\tau'$  which is also strongly  $w$ -admissible (again by Remark 2.3.2).

By construction,  $X_{j,v}$  is a union of cells of  $\mathcal{E}_{j+1}$  and the union of all the closure of the cells of dimension  $(j+1)$  of the barycentric subdivision of  $\mathcal{E}_{j+1}$  which are  $(j+1; \mathbb{N}u)$ -thin is precisely  $Y_{j+1,u}$ . Therefore we may apply Proposition 3.2.1 to  $X_v$ . This provides a map  $\varphi : \widehat{C}_j^w(X_v; \mathcal{F}) \rightarrow \widehat{C}_j^u(Y_{j+1,u}; \mathcal{F})$  such that

$$\partial\varphi_{\tau'} = \varphi_{\partial\tau'} = \varphi_{\sigma'}.$$

As by (iii) of this proposition  $\sigma' = \varphi_{\sigma'}$  in  $H_j^u(X_{j,v})$ , this implies that the class of  $\sigma$  is zero in  $H_j^u(Y_{j+1,u})$  and yields that  $b''$  is one-to-one.  $\square$

**Corollary 3.2.3.** *For any closed definable subset  $X$ , the vanishing homology groups  $H_j^v(X)$  are finitely generated.*

Note that the above corollary enables us to define an Euler characteristic which is a definable metric invariant by setting:

$$\chi_v(X) := \sum_{i=1}^{\infty} (-1)^i \dim H_i^v(X).$$

This invariant for definable subsets of  $R^n$  gives rise to a metric for definable families or for germs of definable sets (see example 1.3.2 and section 4 below).

**Remark 3.2.4.** The hypothesis closed is assumed for convenience. We could shrink an open tubular neighborhood of radius  $z \in v$  of the points lying in the closure but not in  $X$  so that we would have a deformation retract of our set onto the complement of this neighborhood which is very close to the identity, and hence which preserves thin subsets, identifying the vanishing homology groups of our given set with those of a closed subset.

#### 4. LOCAL INVARIANTS FOR SINGULARITIES.

In [V2], we introduced the link for a semi-algebraic metric space. Let us recall its definition. We recall that we denote by  $k(0_+)$  the field of algebraic Puiseux series endowed with the order that makes the indeterminate  $T$  positive and smaller than any real number. We denote by  $d$  the Euclidian distance. Given the germ at 0 of a semi-algebraic set  $X$  let:

$$L_X := \{x \in X_{k(0_+)} : d(x; 0) = T\}$$

where  $T \in k(0_+)$  is the indeterminate and  $X_{k(0_+)}$  the extension of  $X$  to  $k(0_+)$ .

In [V2] we proved that the set  $L_X$  is a metric invariant which characterizes the metric type of the singularity in the sense that:

**Theorem 4.0.5.** [V2]  $L_X$  is semi-algebraically bi-Lipschitz homeomorphic to  $L_Y$  iff  $Y$  is semi-algebraically bi-Lipschitz homeomorphic to  $X$ .

This theorem admits the following immediate corollary.

**Corollary 4.0.6.** For any convex subgroup  $v \subset k(0_+)$ , the groups  $H_j^v(L_X)$  are semi-algebraic bi-Lipschitz invariants of  $X$ .

Note that by Corollary 3.2.3 these groups are finitely generated and that  $\chi_v(L_X)$  is a semi-algebraic bi-Lipschitz invariant of the germ  $X$ .

**Remark 4.0.7.** We assumed in this section that  $X$  is a semi-algebraic set because this was the setting of [V2]. Nevertheless, the main ingredient of the proof of Theorem 4.0.5 is Theorem 5.1.3 of [V1]. As this theorem holds over any polynomially bounded o-minimal structure, the above corollary is still true in this setting as well. The metric type of the link  $L_X$  may fail to be a metric invariant of the singularity when the set is definable in a non-polynomially bounded o-minimal structure as it is shown by the following example.

**Example 4.0.8.** Let  $X := \{(x; y) \in \mathbb{R}^2 : |y| = e^{\frac{-1}{x^2}}\}$  and  $Y = \{(x; y) \in \mathbb{R}^2 : |y| = e^{\frac{-2}{x^2}}\}$ . Note that  $X$  and  $Y$  are both definable in the  $ln - exp$  structure (see [vDS], [LR], [W]). Furthermore  $X$  and  $Y$  are definably bi-Lipschitz homeomorphic. However the links of  $X$  and  $Y$  are constituted by two points of  $k_{0+}^2$  (where  $k_{0+}$  is the corresponding residue field) whose respective distances are clearly not equivalent.

Note that a revolution of these subsets about the  $x$ -axis provides two subsets whose links have different vanishing homology groups (for a suitable velocity).

## 5. SOME EXAMPLES.

We give two examples of computations of the homology groups. It is convenient to develop ad hoc techniques to compute the homology groups such as the excision property.

**5.1. The excision property.** It follows from the definition that we may have  $c + c'$  in  $C_j^v$  although neither  $c$  nor  $c'$  belong to this set. This is embarrassing since it makes it impossible the splitting of a chain of  $X$  into a chain of  $X \setminus A$  plus a chain of  $A$ , which is crucial for the excision property. To overcome this difficulty we are going to consider more chains. This will *not* affect the resulting homology groups.

We defined the vanishing homology groups by requiring for a chain  $\sigma$  that  $|\sigma|$  and  $|\partial\sigma|$  to be both  $(j; v)$ -thin. We may work with another chain complex.

Let  $A$  and  $X$  be closed definable subsets of  $R^n$  with  $A \subset X$  and denote by  $\mathcal{F}$  the pair  $\{X \setminus \text{Int}(A); A\}$  where  $\text{Int}(A)$  is the interior of  $A$ . Let  $\Delta_j^v(X)$  the subset of  $C_j^v(X; \mathcal{F})$  constituted by the  $j$ -chains having a  $(j; v)$ -thin support. Of course, such a family of modules is not preserved by the boundary operator but, if we want to have a chain complex, we may add the boundaries by setting:

$$\Delta_j'^v(X) := \Delta_j^v(X) + \partial\Delta_{j+1}^v(X).$$

This provides a chain complex with obviously  $H_j(\Delta'^v(X)) = H_j^v(X)$ .

The inconvenient is that we are going to work with non admissible chains but the advantage is that we have now more freedom to work since we have more chains. For instance if  $(c_1 + c_2) \in \Delta_j^v(X)$  then  $c_1$  and  $c_2$  both belong to  $\Delta_j^v(X)$ .

To state the excision property we need to introduce the homology groups of a pair. For this purpose, we first set:

$$\Delta_j^v(X/A) := \{c \in \Delta_j^v(X) : (\partial c - \partial_A c) \in \Delta_{j-1}^v(X)\},$$

where  $\partial_A$  takes the boundary and projects it onto  $C_j(A)$ .

Define also

$$\Delta_j^{v,X}(A) := \Delta_j^v(A) + \partial_A \Delta_{j+1}^v(X/A).$$

First observe that by definition if  $c \in \Delta_{j+1}^v(X/A)$  then

$$\partial_A c \in \Delta_j^v(X) + \partial \Delta_{j+1}^v(X).$$

Therefore, by definition of  $\Delta_j^{v,X}$  we get

$$\Delta_j^{v,X}(A) \subset \Delta_j^v(X) + \partial \Delta_{j+1}^v(X) = \Delta_j^v(X).$$

Thus, we may set

$$\Delta_j^v(X; A) := \frac{\Delta_j^v(X)}{\Delta_j^{v,X}(A)},$$

and

$$H_j^v(X; A) := H_j(\Delta_j^v(X; A)).$$

**Remark 5.1.1.** If  $X$  is a  $v$ -thin set of dimension  $j$  then  $H_j^v(X; A) = H_j(X; A)$  (see Remark 1.3.1).

Let  $i : \Delta_j^{v,X}(A) \rightarrow \Delta_j^v(X)$  be the inclusion. Clearly, we have the following exact sequences:

$$(5.15) \quad 0 \rightarrow \Delta_j^{v,X}(A) \xrightarrow{i} \Delta_j^v(X) \xrightarrow{q} \Delta_j^v(X; A) \rightarrow 0,$$

(where  $q$  is the quotient map) and therefore we get the following long exact sequence:

$$\dots \rightarrow H_j^v(\delta_X A) \rightarrow H_j^v(X) \rightarrow H_j^v(X; A) \rightarrow H_{j-1}^v(\delta_X A) \rightarrow \dots$$

**Remark 5.1.2.** We could have defined the homology groups of a pair by  $H_j^v(X; A) := H_j^v(C^v(X; A))$  where  $C^v(X; A) := \frac{C_j^v(X)}{C_j^v(A)}$ , and of course the latter exact sequence would hold for  $H_j^v(A)$  (instead of  $H_j^v(\delta_X A)$ ). However the excision property would not hold.

As we said, if  $(c + c')$  belongs to  $\Delta_j^v(X)$  then  $c$  and  $c'$  both belong to  $C_j^v(X)$ . Therefore, the excision property holds for  $H_j^v(X; A)$ . Let  $(X; A)$  and  $W$  be definable such that  $W$  lies in the interior of  $A$ . Then for any  $j$ :

$$(5.16) \quad H_j^v(X; A) = H_j^v(X \setminus W; A \setminus W).$$



**5.2. Two examples.** We are going to compute the vanishing homology groups on two examples which are semi-algebraic families. Let us take  $\mathbb{Q}$  as our coefficient group.

**Example 5.2.1.** We first compute the homology groups on the example sketched on fig 1. We consider two spheres from which we shrink a little disk which collapses into a point and which intersects along the boundaries of these disks.

Let

$$X(\varepsilon) := \{(x; y; z) \in k(0_+)^3 : (x - \varepsilon(1 - T^4))^2 + y^2 + z^2 = 1, \varepsilon x \geq 0\}$$

for  $\varepsilon = \pm 1$ . Then let  $X := X(1) \cup X(-1)$  and  $A = X(1) \cap X(-1)$ .

Let us simply consider the velocity  $T^2$ . The computation could actually be carried out for any velocity. Since the set  $A$  is  $NT^2$ -thin we have:

$$H_1^{T^2}(\delta_X A) = H_1^{T^2}(A) = H_1(A) = \mathbb{Q},$$

and  $H_0^{T^2}(\delta_X A) = 0$ .

Note that, thanks to the excision property, we have:

$$H_1^v(X; A) \simeq H_1^v(X(1); A) \oplus H_1^v(X(-1); A).$$

If we add the disk

$$D = \{(x; y; z) \in k(0_+)^3 : (x - \varepsilon(1 - T^4))^2 + y^2 + z^2 = 1, \varepsilon x \leq 0\}$$

to  $X(1)$ , we get the sphere  $S^2$ . Thus, by the excision property,

$$H_1^{T^2}(X(1); A) \simeq H_1^{T^2}(S^2; D) = 0,$$

and so  $H_1^{T^2}(X; A) = 0$ . Examining the exact sequence of the pair  $(X; A)$  we see that:

$$H_1^{T^2}(X) \simeq H_1^{T^2}(\delta_X A) \simeq \mathbb{Q}.$$

Observe also that we have:  $H_2^{T^2}(X) \simeq 0$  and  $H_0^{T^2}(X) \simeq 0$ .

We end by computing the vanishing homology groups of Birbrair-Goldshtein examples (compare with [BB1] section 7.) .

**Example 5.2.2.** Let  $X$  be the set defined by (0.2) assume that  $p < q$ . Let us compute for instance the vanishing homology groups for the velocity  $T^q$ . We could use here the excision property and follow the classical methods for computing the homology groups of the torus but it is actually simpler to derive it from the classical homology groups of  $X$  since it is  $NT^q$ -thin. This implies that the inclusion  $H_2^{T^q}(X) \rightarrow H_2(X)$  is an isomorphism and that the inclusion  $H_1^{T^q}(X) \rightarrow H_1(X)$  is one-to-one. Therefore

$$H_2^{T^q}(X) \simeq \mathbb{Q}$$

and  $\dim H_1^{T^q}(X) \leq 2$ . Actually, one generator of  $H_1(X)$  has a representant with  $T^q$ -thin support and every 1-chain representing a different class has a support whose length is clearly bigger than  $T^p$ . This proves that  $\dim H_1^{T^q}(X) = 1$ .

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